Spectroscopy from the lattice:
the scalar glueball and $K\pi$ scattering

Ruairí Brett
Carnegie Mellon University

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Motivation

- significant experimental efforts to map out QCD spectrum: Hall B/D (JLab), COMPASS (CERN), etc.

- few states (QCD)-stable: $\pi, K, N, \Lambda, \Sigma, \Xi, \Omega$
  - most are unstable resonances

- theoretical description of states important, many poorly understood
  - eg. tetraquarks ($qq\bar{q}\bar{q}$), glueballs (bound gluons), …

- study low-lying hadronic spectrum using lattice QCD:
  - QCD stationary states in finite-vol.
  - Lüscher: $2 \rightarrow 2$ scattering amplitudes from finite-vol. energies
QCD on the Lattice

- strong coupling, $\alpha_s$, large at low energies
  → non-perturbative

\[\alpha_s(M_z) = 0.1181 \pm 0.0011\]

\[pp \rightarrow \text{jets} \quad \text{e.w. precision fits (N}^3\text{LO)}\]

\[Q \quad \text{[GeV]} \quad 1 \quad 10 \quad 100\]

\[\tau \text{ decays (N}^3\text{LO)}\]

\[\text{DIS jets (NLO)}\]

\[\text{Heavy Quarkonia (NLO)}\]

\[e^+e^- \text{ jets & shapes (res. NNLO)}\]

\[\text{e.w. precision fits (N}^3\text{LO)}\]

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\[\text{Particle Data Group}\]
QCD on the Lattice

- strong coupling, $\alpha_s$, large at low energies
  $\rightarrow$ non-perturbative

- Lattice QCD: define theory on finite, Euclidean, space-time lattice
  - lattice spacing $\rightarrow$ regulator

- observables:
  $$\langle A \rangle = \frac{1}{Z} \int \mathcal{D}[\bar{\psi}, \psi, U] A e^{-S[\bar{\psi}, \psi, U]}$$

- evaluate numerically via Monte Carlo
  $\Rightarrow$ control over (many) systematic errors, etc.
Rotational Symmetry

- periodic B.C. in cubic box:
  \[ J^{PC} \] no longer good quantum numbers

- label stationary states using irreps of cubic symmetry group \( O_h \)
  - parity (\( u/g \)) and G-parity (\(+/-\)) where relevant

- continuum spin ID:

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<th>( A_2 )</th>
<th>( E )</th>
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Extracting the Finite-Volume Spectrum

- temporal correlation matrix:

\[ C_{\alpha\beta}(t) \equiv \langle 0|O_{\alpha}(t + t_0)O_{\beta}(t_0)|0 \rangle \]

\[ = \sum_n \langle 0|O_{\alpha}|n\rangle \langle n|O_{\beta}|0 \rangle e^{-E_nt} \]

- eigenvalues tend to lowest \( N \) energies

\[ \lim_{t \to \infty} \lambda_n(t) = b_n e^{-E_nt}[1 + O(e^{\Delta_n t})] \]

\[ \Rightarrow E_{\text{eff}}^n(t) = \frac{1}{\Delta t} \ln \left( \frac{\lambda_n(t)}{\lambda_n(t + \Delta t)} \right) \]
Extracting the Finite-Volume Spectrum

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quality of signal.excited state contamination \textit{highly} dependant on operator basis
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quality of signal/excited state contamination *highly* dependant on operator basis

careful operator design is crucial
Extracting the Finite-Volume Spectrum

- 2-pt. correlation matrix:

\[ C_{\alpha\beta}(t) = \sum_n \langle 0 | O_\alpha | n \rangle \langle n | \overline{O}_\beta | 0 \rangle e^{-E_n t} \]

\[ = \sum_n Z^{(n)}_\alpha Z^{(n)*}_\beta e^{-E_n t} \]

- level ID inferred from \( Z \) overlaps with \textit{probe} operators:

\[ |\Phi_j\rangle \equiv O_j |0\rangle \Rightarrow Z^{(n)}_j = \langle \Phi_j | n \rangle \]

multiple operators can mix/overlap on any given eigenstate
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multiple operators can mix/overlap on any given eigenstate

overlaps give **qualitative** measure of mixing between states
The Scalar Glueball

- glueball: hypothetical bound state of gluons

- experimental evidence elusive, light scalar candidates:
  - \( f_0(1370), f_0(1500), f_0(1710) \)

- lattice studies to date:
  - light scalar \( \sim 1700 \text{ MeV} \)
  - most in pure \( SU(3)/\text{quenched} \) approx. (no quark/meson mixing)
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- here: extract low-lying $A_{1g}^+$ spectrum with $q\bar{q}$, meson-meson, & glueball operators
  - first look (from the lattice) at mixing between glueball, $q\bar{q}$, and two-hadron states

*PRD 73, 014516 (2006)*
A$_{1g}^+$ Spectrum

$24^3 \times 128$ anisotropic lattice, $m_\pi \sim 390$ MeV, $m_K \sim 550$ MeV:

![Graphs showing energy levels and significances with and without glueball operators.]

- $m_\pi \sim 390$ MeV
- Hatched boxes: significant overlap with multiple operators
- $m_{\text{ref}} = 1.82 m_K \sim 1$ GeV
Scattering process: eg.

\[ I = 1 \quad \pi\pi \rightarrow \pi\pi \]

\[ \infty \text{-volume} \]

\[ 2m_\pi \]

bound states

meson-meson continuum

\[ E \]
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\[ \begin{array}{c}
\text{bound states} \\
\text{meson-meson continuum}
\end{array} \]

\[ 2m_\pi \]

\[ E \]

\[ \text{finite volume} \]

\[ 2m_\pi \]

\[ E \]

no continuum of scattering states \( \rightarrow \) how to access \( \infty \)-vol. physics?

\[ p = \frac{2\pi}{L} d \]
Lüscher Quantisation \((2 \rightarrow 2)\)

Infinite volume physics from LQCD

Lüscher: Relationship between finite volume spectrum and infinite volume scattering matrix

- Quantisation condition:

\[
\det[\tilde{K}^{-1} - B] = 0
\]

- For each \(E_{\text{cm}}\) in spectrum, determinant gives single relation to entire scattering matrix

  ⇒ Exactly solvable for single channel, single partial wave

  ⇒ \(\ell\) mixing/coupled decay channels requires parameterisation of \(\tilde{K}\) and a fit

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\]

\text{box matrix: known function of } (E_{cm}, L)

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\[\Rightarrow\] \(\ell\) mixing/coupled decay channels requires parameterisation of \(\tilde{K}\) and a fit

Elastic $K\pi$ Scattering

- $32^3 \times 256$ anisotropic lattice,
  $m_\pi \approx 230$ MeV
- include $\ell = 0, 1, 2$ partial waves
- fit forms

$$
\left( \tilde{K}^{-1} \right)_{00}^{\text{ERE}} = \frac{-1}{m_\pi a_0} + \frac{m_\pi r_0}{2} \frac{q_{\text{cm}}^2}{m_\pi^2}
$$

$$
\left( \tilde{K}^{-1} \right)_{11}^{\text{BW}} = \frac{6\pi E_{\text{cm}}}{g^2 m_\pi} \left( \frac{m_{K^*}^2}{m_\pi^2} - \frac{E_{\text{cm}}^2}{m_\pi^2} \right)
$$

$$
\left( \tilde{K}^{-1} \right)_{22}^{\text{ERE}} = \frac{-1}{m_\pi^5 a_2}
$$

- $p$-wave : $K^*(892)$
- $s$-wave : $K_0^*(700)/\kappa$

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$K\pi$ Scattering Phase-Shifts

- plots of $p$-wave and $s$-wave phase shift ($\tilde{E} = (E_{cm} - m_K)/m_\pi$)
- $d$-wave contributes negligibly
Decay of $K^*(892)$

- comparison with (some) lattice calculations of $K^*(892)$ resonance parameters
- phenomenological values shown with asterisks
Conclusions/Future work - Scattering

- goal: calculate scattering phase-shifts/resonance parameters from LQCD

- meson-meson scattering at a mature stage
  - see B. Hörz’s talk today
  - physical point calculations feasible - large volumes required

  - see J. Bulava’s plenary this morning
  - baryon-baryon at light quark masses on the horizon

- extension of formalism to include three-body channels underway
  - see M. Mai’s talk on Mon.
- goal: *qualitative* description of resonance spectrum

- high computational cost for large operator bases/volumes
  - careful operator design crucial

- $\bar{q}q$ states straightforward but many interesting states not well described
  $\Rightarrow$ hybrids
  $\Rightarrow$ molecular states
  $\Rightarrow$ ...

- influence of tetraquark operators: $\kappa$ and $a_0(980)$
  - see D. Darvish’s talk on Tues.
Meson configurations

- $q$'s = smeared, displaced quark fields

\[
\Phi^{AB}_{\alpha\beta;ij}(p, t) = \sum_x e^{ip \cdot (x + \frac{1}{2}(d_\alpha + d_\beta))} \delta_{ab} q^{B}_{b\beta j}(x, t) q^{A}_{a\alpha i}(x, t)
\]

\[
\Phi^{ABC}_{\alpha\beta\gamma;ijk}(p, t) = \sum_x e^{ip \cdot x} \varepsilon_{abc} q^{C}_{c\gamma k}(x, t) q^{B}_{b\beta j}(x, t) q^{A}_{a\alpha i}(x, t)
\]

- group-theory projections onto irreps of lattice symmetry group

\[
\Phi_{l}(t) = c^{(l)*}_{\alpha\beta} \Phi^{AB}_{\alpha\beta}(t) \quad \Phi^{l}_{l}(t) = c^{(l)*}_{\alpha\beta\gamma} \Phi^{ABC}_{\alpha\beta\gamma}(t)
\]
More exotic operators

- tetraquarks: $3 \times \overline{3} \times \overline{3} \times \overline{3} = 1 + 1 + 8 + 8 + \ldots$

$$\Phi_{\alpha\beta\mu\nu;ijkl}^{ABC\bar{D}}(p, t) = \sum_x e^{ip \cdot x} (\delta_{ab}\delta_{cd} \pm \delta_{ad}\delta_{bc})$$

$$\times \bar{q}_d^{D}(x, t) \ q_c^{C}(x, t) \ \bar{q}_b^{B}(x, t) \ q_a^{A}(x, t)$$

→ preprint coming soon: $\kappa$ & $a_0(980)$ resonances
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**More exotic operators**

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\]

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- scalar glueball: (purely gluonic - no quark fields)

\[ G_\Delta \sim \text{Tr} \, \tilde{\Delta}, \quad \tilde{\Delta} \equiv \text{covariant laplacian} \]

→ see PRD 88, 014511 (2013)
\( S\)-wave \( K\pi \) amplitude: \( K_0^*(700)/\kappa \)

\[
\frac{m_{K^*}}{m_\pi} = 3.808(18), \quad g = 5.33(20), \quad m_\pi a_0 = -0.353(25), \\
m_\pi^5 a_2 = -0.0013(68), \quad \chi^2/\text{dof} = 1.42
\]

- based on LO ERE, \( m_\pi a_0 < 0 \) suggests virtual bound state

- however, NLO parameters give \( 1 - 2r_0/a_0 = -8.9(2.4) \) which must be \( > 0 \) for a (real or virtual) bound state

- zeros of \( q_{\text{cm}} \cot \delta_0 - iq_{\text{cm}}: m_R/m_\pi = 4.66(13) - 0.87(18)i \)
  - consistent with BW fit

- better energy resolution & careful analytic continuation required

[\text{Nucl.Phys. B932 (2018)}]