Pion–pion scattering and the timelike pion form factor from Lattice QCD

Ben Hörz (LBNL)

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However better knowledge of contributions from heavier resonances is also important for higher energy experiments like NOvA and DUNE and seriously lacking. The most important challenges are

- improving our knowledge of the axial part of nucleon-Δ transition matrix elements, either via a new hydrogen and/or deuterium experiment or via lattice-QCD calculations;
- describing nonresonant contributions to pion production channels. Understanding the range

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\langle N \pi | J_\mu | N \rangle
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Salient Features of Lattice QCD

- lattice spacing $a$
- spatial extent $L$
- Euclidean time

**basic observable: correlation functions**

$$C_{ij}(t) = \langle 0 | O_i(t) \, \bar{O}_j(0) | 0 \rangle$$

$$= \sum_n \langle 0 | O_i | n \rangle \langle n | \bar{O}_j | 0 \rangle \, e^{-E_n t}$$

How do we get **scattering** from **finite-volume** observables?

[see also talk by M. Mai earlier this week]
Scattering from Lattice QCD

single particle in a periodic box
\[ \Delta E \propto e^{-mL} \]

two spinless particles in a periodic box
\[ \Delta E \propto a_0/L^3 + O(L^{-4}) \]

\[ L \]

⇒ ‘The Lüscher method’

[Lüscher ’86, ’91]
Scattering processes and resonances from lattice QCD

Raúl A. Briceño, 1,* Jozef J. Dudek, 1,2, † and Ross D. Young 3, ‡

1 Thomas Jefferson National Accelerator Facility, 12000 Jefferson Avenue, Newport News, Virginia 23606, USA
2 Department of Physics, College of William and Mary, Williamsburg, Virginia 23187, USA
3 Special Research Center for the Subatomic Structure of Matter (CSSM), Department of Physics, University of Adelaide, Adelaide 5005, Australia

(Dated: June 21, 2017)

The vast majority of hadrons observed in nature are not stable under the strong interaction, rather they are resonances whose existence is deduced from enhancements in the energy dependence of scattering amplitudes. The study of hadron resonances of-
Two-particle Quantization Condition

\[ \det \left[ \mathcal{M}^{-1}(E_L) + F(E_L, L) \right] = 0 \]

- \(E_L\) – FV spectrum
- \(\mathcal{M}\) – 2-to-2 scatt. ampl.
- \(F\) – known functions

2-particle channel
partial wave
(total angular mom.)
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group theory worked out and publicly available on Github

[Morningstar, Bulava, Singha, Brett, Fallica, Hanlon, BH 1707.05817]
A SIMPLE (YET RELEVANT) RESONANCE: $\rho(770)$

[plot adapted from Bulava, Fahy, BH, Juge, Morningstar, Wong 1604.05593]

- elastic $\pi\pi$ scattering neglecting $\ell \geq 3$ partial wave spectrum $\Leftrightarrow$ scattering amplitude

- benchmark system for the lattice

  e.g. Lang et al. 1105.5636, Aoki et al. 1106.5365, ..., Dudek et al. 1212.0830, ...

- recent interest due to its contribution to $(g-2)_\mu$ HVP

  [Meyer, Wittig 1807.09370]
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Matrix elements: Timelike pion form factor

- Muon anomalous magnetic moment \((g - 2)_\mu\)
  - HVP governed by
    \[
    R_{\text{had}} = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\frac{4\pi \alpha_{\text{em}}(s)^2}{3s}}
    \]
  - Two-pion state dominates at low energies
    \[
    R_{\text{had}}(s) = \frac{1}{4} \left(1 - \frac{4m_{\pi}^2}{s}\right)^{\frac{3}{2}} |F_\pi(s)|^2
    \]

\(\gamma^* \rightarrow \pi\pi\)
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\[
|F_{\pi}(E^*)|^2 = g_\Lambda(\gamma) q \frac{\partial(\delta_{1+F})}{\partial q} \frac{3\pi E^{*2}}{2q^5 L^3} \left| \langle 0 | V^{(d,\Lambda)} | d\Delta E^* \rangle \right|^2
\]

Infinite volume

Finite volume

Requires scattering amplitude

Lellouch, Lüscher hep-lat/0003023
Meyer 1105.1892
Feng et al. 1412.6319
Vector-correlator reconstruction

\[ \frac{5}{9} G_l(t) \bar{K}(t) / m_\mu \]

- time-momentum representation
  \cite{Bernecker, Meyer 1107.4388}

- clear spectral decomposition

\[ G_l(t) \sim \sum_x \langle 0| V(x, t) V^\dagger(0)|0 \rangle \]

\[ \approx \sum_n \left| \langle 0| V|T_{1u}, n \rangle \right|^2 e^{-E_n t} \]

\[ m_\pi = 200 \text{ MeV}, \quad L = 4.1 \text{ fm} \]

\cite{Gérardin, Cè, von Hippel, BH, Meyer, Mohler, Ottnad, Wilhelm, Wittig 1904.03120}
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\[ \frac{5}{9} G_l(t) \overline{K}(t)/m_\mu \]

Light

N=3

N=2

N=1

\[ t \text{ [fm]} \]

\[ m_\pi = 200 \text{ MeV}, L = 4.1 \text{ fm} \]

[Bernecker, Meyer 1107.4388]
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[Bounding method]

- Improved bounding method
Timelike pion form factor results

\[ |F_{\pi}| \]

\( E_{cm}/m_\pi \)

\( m_\pi = 260 \text{ MeV} \)

\( m_\pi = 280 \text{ MeV} \)

\( a = 0.076 \text{ fm} \)

\( a = 0.064 \text{ fm} \)

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\( \Rightarrow \) formalism and technology under control for two-meson systems

[Andersen, Bulava, BH, Morningstar 1808.05007]
Multi-hadron spectroscopy

$\pi\pi \ I = 2$

$\pi\pi\pi \ I = 3$

$m_\pi = 200 \text{ MeV}, \ m_\pi L = 4.2$

- significant energy shifts in three-pion ground and excited states
  (remember: energy shifts encode interaction)

- finite-volume formalism to reveal three-particle interaction
  much development in recent years

Sharpe, Hansen, Briceño, Hammer, Rusetsky, Polejaeva, Mai, Döring, ...

[BH, Hanlon 1905.04277]

[Hansen, Sharpe 1901.00483]
Summary & Outlook

• hadron interactions from lattice QCD ...
  (calculations maturing, starting to assess standard lattice systematics)

• ...and with practical relevance
  (helping to improve \((g-2)_\mu /HVP\) from lattice QCD)

• similar method for access to \(\Delta\) transition form factor
  (for \(N\pi\) scattering results see talk by J. Bulava)

• stay tuned for
  • \(NN\) scattering
  • \(N\Sigma, N\Lambda\) scattering
  • ...

[11/11]