Quark, gluon and hadron physics within a novel renormalization-group procedure for the QCD Hamiltonian

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   The method of calculation

2. Effective particles in QCD
   Effective quarks and gluons

3. Effective interactions in hadronic reactions
   The scattering problem
Motivation

What are natural scales in a problem?
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**Fundamental theories**

Degrees of freedom are given and they are infinite

\[ H_{\text{QCD}} |\psi\rangle = E |\psi\rangle \]

\[ H_{\text{QCD}} = H_q + H_{\bar{q}} + H_g + H_{Qqg} + H_{q\bar{q}} + \ldots \]

\[ |\psi\rangle = |0\rangle + |q\rangle + |\bar{q}\rangle + |g\rangle + |qg\rangle + |q\bar{q}\rangle + \ldots \]

Local interactions produce divergences \( \int_{\infty} \)

**Effective theories**

\[ \rightarrow \text{Tool: Głazek-Wilson Similarity Renormalization Group (SRG):} \]

\[ H_\lambda = U_\lambda H_0 U_\lambda^\dagger, \quad U_\lambda |\psi_0\rangle = |\psi_\lambda\rangle, \quad H_\lambda |\psi_\lambda\rangle = H_0 |\psi_0\rangle \]

Then, truncation of the Fock space makes sense for low energies

Cutoff \( \int^\Lambda \rightarrow \frac{\partial [\text{Observables}]}{\partial \Lambda} = 0, \quad \Lambda \sim \Lambda_{\text{natural}}. \)
Effective-particle approach

Renormalization group procedure for effective particles (RGPEP)

★ Derive the canonical Hamiltonian of the theory:
Lagrangian $\rightarrow$ Hamiltonian $\rightarrow$ bound state problem
$\mathcal{L}_{QCD} \rightarrow \mathcal{H}_{QCD} \rightarrow H_{QCD}|\Psi\rangle = E|\Psi\rangle$

★ Construct the effective Hamiltonian:

$$H_{\infty}(q_{\infty}) = H_{\lambda}(q_{\lambda}), \quad H_{\lambda} = U_{\lambda} H_0 U_{\lambda}^\dagger$$

★ Hadrons in the Fock space

$$|\Psi_{s\text{ meson}}\rangle = |Q\lambda \bar{Q}_{\lambda}\rangle + |Q\lambda \bar{Q}_{\lambda} G_{\lambda}\rangle + |Q\lambda \bar{Q}_{\lambda} G_{\lambda} G_{\lambda}\rangle + ...$$

Hadrons as bound states of QCD...
... from asymptotic freedom to bound states

K.G. Wilson et al PRD49 (1994) 6720
The method of calculation

Start from the Lagrangian density $\mathcal{L}_{QCD} = \bar{\psi}(i\slashed{D} - m)\psi - \frac{1}{2} \text{tr} F^{\mu\nu} F_{\mu\nu}$

1. **Canonical Hamiltonian** Use front-form dynamics:

   - $\mathcal{L}_{QCD} \rightarrow T_{QCD}^{\mu\nu} \rightarrow H_{QCD} = \int_{x^+ = 0} \mathcal{H}_{QCD}(x) dx, \quad A^+ = 0$
   - $k^+ = k^0 + k^3, \quad k^- = k^0 - k^3, \quad \vec{k}^\perp = (k^1, k^2)$
   - $x_i = k^+_i / P^+ \quad \kappa^\perp_{ij} = x_j k^\perp_i - x_i k^\perp_j$
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2. **Regularization** Introduce regulating functions at vertices

   - UV and small-$x$ cutoff $\int dx d^2 \kappa^\perp \rightarrow \int dx d^2 \kappa^\perp r_\delta(x) r_\Delta(\kappa^\perp)$
   - $\lim_{\delta \rightarrow 0} r_\delta(x) = 1, \quad \lim_{\Delta \rightarrow \infty} r_\Delta(\kappa^\perp) = 1$
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   $k^+ = k^0 + k^3$, $k^- = k^0 - k^3$, $\vec{k}^\perp = (k^1, k^2)$ ;

   $x_i = k_i^+/P^+$, $\kappa_{ij}^\perp = x_j k_i^\perp - x_i k_j^\perp$

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3. **Renormalization**

   $q^\dagger |0\rangle = |q\rangle \rightarrow \ q^\dagger_\lambda |0\rangle = |q_\lambda\rangle \quad q^\dagger_\lambda = U_\lambda q^\dagger U^\dagger_\lambda$

   Effective particles of type $\lambda$ introduced by RGPEP

   $$ \frac{d}{d\lambda^{-4}} H_\lambda = [G_\lambda, H_\lambda] \ , \quad G_\lambda = [H_f, H_{P\lambda}]$$
Effective quanta

Hamiltonian can be re-written in terms of effective quanta

\[ H_0(q_0) = H_s(q_s) \]

\[ q_0^\dagger|0\rangle = |q_0\rangle \quad \rightarrow \quad q_s^\dagger|0\rangle = |q_s\rangle \quad q_s = U_s q_0 U_s^\dagger \]

\( s = size \)
\( \lambda = 1/s \) momentum scale

RGPEP equation

\[ H_t' = [G_t, H_t] , \quad G_t = [H_{free}, \tilde{H}_t] , \quad t = 1/\lambda^4 \]

Initial condition

\[ H_{\lambda=\infty}(= H_{s=0}) = H_{\text{canonical}}^{\text{QCD}} + CT_{\Delta \delta} \]

Counterterms \( CT^{\Delta \delta} \) remove UV-cutoff \( \Delta \) dependence.
Solve the RGPEP equation perturbatively

\[ H_\lambda(q_\lambda) = H(q) \]

\[ H'_\lambda = [[H_f, H_{P\lambda}], H_\lambda] \quad q_\lambda = U_\lambda q U_\lambda^\dagger \]

perturbatively, order by order

\[ H_\lambda = H_f + gH_{1,\lambda} + g^2H_{2,\lambda} + g^3H_{3,\lambda} + g^4H_{4,\lambda} + ... \]

\[ H'_f = 0 , \]
\[ gH'_{\lambda 1} = [[H_f, gH_{1\lambda}], H_f] , \]
\[ g^2H'_{\lambda 2} = [[H_f, g^2H_{2\lambda}], H_f] + [[H_f, gH_{1\lambda}], gH_{1\lambda}] , \]
\[ g^3H'_{\lambda 3} = [[H_f, g^3H_{3\lambda}], H_f] + [[H_f, g^2H_{2\lambda}], gH_{1\lambda}] + [[H_f, gH_{1\lambda}], g^2H_{2\lambda}] , \]

\[ \rightarrow \text{Integration produces functions with } \text{form factors} \]

\[ e^{-\left( M_a^2 - M_b^2 \right)^2 / \lambda^4} \]
The idea of effective particles

Effective particles of type $\lambda$ can change their relative motion kinetic energy through a single effective interaction by no more than about $\lambda$

$$s_c \sim 1/\Lambda_{QCD}$$

$$f_\lambda = e^{-\left(M_1^2 - M_2^2\right)^2 / \lambda^4} = e^{-\left(M_1^2 - M_2^2\right)^2 s^4}$$

$$s \ll s_c \quad s < s_c \quad s \sim s_c$$
Examples of terms in $H_{\lambda QCD} = H_f + g H_{1,\lambda} + g^2 H_{2,\lambda} + g^3 H_{3,\lambda} + g^4 H_{4,\lambda} + \ldots$

0-th order terms

1-st order terms

2nd-order terms

3rd-order terms

4th-order terms

...
Example of 3rd-order calculation

Three-gluon vertex and running coupling $g_s$ in SU(3) Yang-Mills theory

[MGR, Głazek, PRD 92 (2015) 065005]
→ The three-gluon vertex:
Starting from the light-front $H_{QCD}$, solve RGPEP perturbatively up to 3rd order

$$Y_\lambda = gH_{1\lambda} + g^3H_{3\lambda}$$

$$= \sum_{123} \int [123] \tilde{Y}_\lambda(\kappa_{12}^+, \sigma) a^\dagger_{1,\lambda} a^\dagger_{2,\lambda} a_{3,\lambda} + H.c.$$
Heavy quarkonium and triply heavy baryons

[Głazek, MGR, More, Serafin, PLB 773 (2017)]
[Serafin, MGR, More, Głazek, EPJ C78 (2018)]
The front-form (FF) eigenvalue equation

\[ H_\lambda |\Psi_\lambda\rangle = \frac{M^2 + P^{-2}}{P^+} |\Psi_\lambda\rangle \quad \Rightarrow \quad (H_\lambda P^+ - P^{-2}) |\Psi_\lambda\rangle = M^2 |\Psi_\lambda\rangle \]

Remark:

* Eigenvalue is \( M^2 \) in FF instead of \( H |\Psi\rangle = M |\Psi\rangle \) in instant form (IF);

* At large distances: \( U_{\text{eff, FF}} \approx V_{\text{eff, IF}}^2 \)
  Linear potential in IF \( \Rightarrow \) quadratic potential in FF

\[ V_{IF}(r) \sim \sigma r \quad \Rightarrow \quad V_{FF}(r) \sim \sigma^2 r^2 \]

[Trawiński et al. PRD90 (2014) 074017]
Heavy quarkonium

Construct heavy-flavor effective theory

\[
\begin{pmatrix}
H_f + g^2 H_2 & g H_1 \\
 g H_1 & H_f + g^2 H_2
\end{pmatrix}
\begin{pmatrix}
|Q\bar{Q}G\rangle \\
|Q\bar{Q}\rangle
\end{pmatrix}
= 0,
\]

\[
\downarrow \text{RGPEP (2nd order)}
\]

\[
\begin{pmatrix}
H_f + \mu^2 & g H_{1\lambda} \\
g H_{1\lambda} & H_f + g^2 H_{2\lambda}
\end{pmatrix}
\begin{pmatrix}
|Q\lambda \bar{Q}_\lambda G_\lambda\rangle \\
|Q\lambda \bar{Q}_\lambda\rangle
\end{pmatrix}
= 0.
\]

[G\lazek, MGR, More, Serafin, PLB 773 (2017)]

- Effective potential including gluon degrees of freedom explicitly
- Energy of a single quark is infinity
- Gluon-mass ansatz yields a finite eigenvalue in $Q\bar{Q}$ and $QQQQ$
- Effective quark-antiquark potential: Coulomb + Harmonic oscillator
Heavy baryons

Analogously, for baryons

\[
\begin{bmatrix}
\Lambda_{QCD} & \lambda & m_0 \\
\end{bmatrix} \xrightarrow{\Delta \to \infty} \begin{bmatrix}
\mathcal{K} \bar{\kappa}
\end{bmatrix}
\]

\[
\begin{bmatrix}
H_0 + g^2 H_2 & gH_1 \\
gH_1 & H_0 + g^2 H_2
\end{bmatrix} - E \begin{bmatrix}
|3Q\rangle \\
|3\rangle
\end{bmatrix} = 0,
\]

\[\downarrow \text{RGPEP (2nd order)}\]

\[
\begin{bmatrix}
H_{\lambda 0} + \mu_\lambda^2 & gH_{\lambda 1} \\
gH_{\lambda 1} & H_{\lambda 0} + g^2 H_{\lambda 2}
\end{bmatrix} - E \begin{bmatrix}
|3Q_\lambda G_\lambda\rangle \\
|3Q_\lambda\rangle
\end{bmatrix} = 0.
\]

[Serafin, MGR, More, Głazek, EPJ C78 (2018)]

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Some results

[Serafin, MGR, More, Głażek, EPJ C78 (2018)]

Black: PDG masses, \hspace{1cm} \textbf{Blue}: Our calculation

\textbf{Green}: average of many different approaches [MGR, Hilger, Krassnigg, PRD 93 (2016)].
Summary of Part I

- Asymptotic freedom at large $\lambda$
- Energy of a single quark $|Q\rangle$ is infinity
- $Q\bar{Q}$ and $QQQ$ bound states:
  - finite eigenvalue thanks to gluon-mass ansatz
  - harmonic oscillator correction to Coulomb potential
  - in order to include the running of the coupling, 4th-order terms are needed
- Gluon degrees of freedom included explicitly

Outlook

$|\Psi_{\text{hybrid}}\rangle = |Q\lambda\bar{Q}\lambda\rangle + |Q\lambda\bar{Q}\lambdaG\lambda\rangle + ...$

→ Replace the ansatz $\mu^2$ by the true theory: Do $g^4$ terms lead to the same oscillator potential?
Summary of Part I

★ Asymptotic freedom at large $\lambda$
★ Energy of a single quark $|Q\rangle$ is infinity
★ $Q\bar{Q}$ and $QQ\bar{Q}$ bound states:
  - finite eigenvalue thanks to gluon-mass ansatz
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★ Gluon degrees of freedom included explicitly

Outlook

→ Hybrids $|\Psi_{\text{hybrid}}\rangle = |Q_\lambda \bar{Q}_\lambda G_\lambda\rangle + |Q_\lambda \bar{Q}_\lambda G_\lambda G_\lambda\rangle + \ldots$
→ Replace the ansatz $\mu^2$ by the true theory:
  Do $g^4$ terms lead to the same oscillator potential?
The scattering problem and the SRG

In collaboration with E. Ruiz Arriola
Phase shifts $\delta_{II}$
data analysis taken from:
[García-Martín, Kamiński, Peláez,
Ruiz de Elvira, Ynduráin, PRD83(2011)]
Kadyshevsky equation

- It is a 3D-reduction of the BSE
- It enables a relativistic Hamiltonian interpretation for the scattering problem
- Amenable for numerical analysis
- Useful in view of its application to the three-body interaction (i.e. omega decays into 3 pions..., and so on)

\[
t(\vec{p}', \vec{p}, \sqrt{s}) = v(\vec{p}', \vec{p}) + \int \frac{d^3 q}{(2\pi)^3} \frac{v(\vec{p}', \vec{q})}{4E_q^2} \frac{t(\vec{q}, \vec{p}, \sqrt{s})}{\sqrt{s} - 2E_q + i\varepsilon}
\]


Phaseshifts:

\[
\tan \delta_l(p) = -\frac{\pi}{8} p r_l(p, p, \sqrt{s})
\]

\[
r_l(p', p, \sqrt{s}) = v_l(p', p) + \int_0^\infty dq \frac{q^2}{4E_q^2} v_l(p', q) \frac{1}{\sqrt{s} - 2E_q} r_l(q, p, \sqrt{s})
\]
Kadyshevsky equation

Its corresponding Hamiltonian in the center-of-mass system:

\[ H\Psi_{l}(p) \equiv 2E_{p}\Psi(p) + \int dq \frac{q^{2}}{4E_{q}^{2}}\nu_{l}(p,q)\Psi_{l}(q) \]

The homogeneus Kadyshevsky equation reads

\[ H\Psi_{l}(p) = \sqrt{s}\Psi_{l}(p) \]

Energy shift prescription:


For a given momentum grid, e.g.

\[ \int_{0}^{\Lambda} dp f(p) \rightarrow \sum_{n=1}^{N} w_{n} f(p_{n}) \]

\[ p_{n} = \frac{\Lambda_{\text{num}}}{2} [1-\cos(\pi/N(n-1/2))] \]

\[ w_{n} = \frac{\Lambda_{\text{num}}}{2} \sin(\pi/N(n-1/2)) \]

Phase shifts can be obtained as

\[ \delta_{n} = -\pi \frac{P_{n} - p_{n}}{w_{n}}, \quad \text{where} \quad \sqrt{s_{n}} = 2E_{n} = 2\sqrt{P_{n}^{2} + m^{2}_{\pi}} \]
Phase shifts $\delta_{II}$

Orange: energy shift calculation
Blue: Fit, standard prescription
Red: Experiment

$S_0$ wave

$P_0$ wave

$S_2$ wave
Phase shifts

- In order to fit $\pi\pi$ phase shifts up to energies $\sqrt{s} \lesssim 1\text{GeV}$, a very high-energy momentum tail up to $\sqrt{s} \lesssim 10\text{GeV}$ needs to be considered.

Model potential taken from [Mathelitsch and Garzilazo PRC 32 (1985)] for $S_0$, $P_1$ and $S_2$, respectively.

- Disparity in energy scales: annoying
- Details of the interaction at short distances are so relevant??
- Is it possible to construct an effective theory and focus in a short energy range?
**Similarity renormalization group**

**SRG** employs a transformation that changes the cutoff to isolate Hamiltonians that produce cutoff-independent eigenvalues.

**SRG** allows to select the relevant energy scale

The transformation

\[ H_\lambda = U_\lambda H_0 U_\lambda^\dagger \]

does not change the spectrum

\[ H|\psi\rangle = H_\lambda |\psi_\lambda\rangle = E|\psi\rangle \]

Different generators \( G_\lambda \) can be chosen, so that \( H_\lambda \) becomes

- Diagonal
- Block-diagonal
The Crank-Nicolson method

Consider the Kadyshevsky equation

\[ H \Psi_t(p) = \sqrt{s} \Psi_t(p) \]

SRG evolution obeys

\[ H'_t = [G_t, H_t] \]

The effective Hamiltonian is related to the initial one by a unitary transformation:

\[ H_t = U_t H_0 U_t^\dagger \]

The unitary transformation must be such that

\[ \frac{dU_t}{dt} = G_t U_t \equiv -i \mathcal{H} U_t \]

One can apply the Crank-Nicolson algorithm,

\[ U_{n+1} = \left( 1 - i \frac{dt}{2} \mathcal{H}_n \right) \left( 1 + i \frac{dt}{2} \mathcal{H}_n \right)^{-1} U_n \]

with \(-i \mathcal{H} = G_t\).
Effective Hamiltonian matrix

$S0$-wave: Evolution of the Hamiltonian matrix for
$\lambda = \infty$, $\lambda = 0.27$ fm$^{-1}$, $\lambda = 0.18$ fm$^{-1}$, and $\lambda = 0.12$ fm$^{-1}$

Wilson diagonal generator:

⇒ there is hope for finding an effective potential “contained” in the small matrix.
Summary:

- Similarity transformation allows to construct an effective Hamiltonian matrix
- Block-diagonal matrix $\rightarrow$ effective potential $\rightarrow$ possible to eliminate long-momentum tails

$\Rightarrow$ SRG is a useful tool not only in nuclear physics: hadronic reactions and QCD.

OUTLOOK:

$\rightarrow$ $3\pi$ resonant states (e.g. $\omega \rightarrow 3\pi$, $A_1$, etc).

$\rightarrow$ Interactions between other hadrons: hadron molecules
Acknowledgment

Thank you for your attention!

This work is part of a larger project

IMTREPHS

*Implicit transverse renormalization in effective partonic hadron structure*

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